ON THE COHOMOLOGY OF A SIMPLE NORMAL CROSSINGS DIVISOR

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Abstract. We establish a formula which decomposes the cohomologies of various sheaves on a simple normal crossings divisor (SNC) D in terms of the simplicial cohomologies of the dual complex $\Delta(D)$ with coefficients in a presheaf of vector spaces. This presheaf consists precisely of the corresponding cohomology data on the components of D and on their intersections. We use this formula to give a Hodge decomposition for SNC divisors and investigate the toric setting. We also conjecture the existence of such a formula for effective non-reduced divisors with SNC support, and show that this would imply the vanishing of the higher simplicial cohomologies of the dual complex associated to a resolution of an isolated rational singularity.

1. Introduction

The purpose of this paper is to establish and exploit a connection between the simplicial cohomology of topology and the Zariski sheaf cohomology of algebraic geometry, in order to understand the cohomological behavior of certain sheaves on a simple normal crossings (SNC) divisor. To a simple normal crossings divisor $D = \sum D_i$ on a smooth variety, that is one such that each component D_i is smooth and each $D_{i_1} \cap ... \cap D_{i_t}$ is a smooth transverse intersection, one may associate a dual CW complex $\Delta(D)$ using only incidence information, and under the assumption that each $D_{i_0} \cap ... \cap D_{i_p}$ is irreducible, $\Delta(D)$ is actually a simplicial complex. Just think of the prime components D_i as the vertices, non-empty 2-fold intersections $D_i \cap D_j$ as the edges, non-empty 3-fold intersections $D_i \cap D_j \cap D_k$ as the 2-faces, and so on. This simplicial complex, first studied by G. L. Gordon [9], is absent of any algebrao-geometric structure yet carries with it important skeletal information.

In a series of recent papers [16], [17], [18], D. A. Stepanov investigated the homotopy type of the dual complex $\Delta(D)$ associated to a resolution of an isolated singularity (one exists by Hironaka's resolution theorem [13]), which he proves is independent of the choice of resolution (using the weak factorization theorem of birational maps, proven by Włodarczyk in [22] and [23], and by Abramovich-Karu-Matsuki-Włodarczyk in [1]) and is therefore an invariant of the singularity. It is known that the homotopy type of a rational surface singularity is trivial [2], and in this spirit Stepanov shows that $\Delta(D)$ is contractible, i.e. homotopic to a point, for isolated toric singularities, 3-dimensional isolated rational hypersurface singularities, 3-dimensional terminal singularities, and Brieskorn singularities. He asks if the homotopy type of the dual complex associated to an isolated

rational singularity is trivial in higher dimensions, and we study the connection between the simplicial cohomologies of $\Delta(D)$ and $H^i(D, \mathcal{O}_D)$ with this conjecture in mind.

In fact, using a completely different approach involving Berkovich's analytic spaces, A. Thuillier has shown [19] more generally that given any ideal $\mathcal{I} \subset \mathcal{O}_X$ defining a closed subscheme Z of a scheme X over a perfect field and a proper map $f : X' \to X$, with $f^{-1}Z = D$ a normal crossings divisor on X' regular, restricting to an isomorphism $X' - D \cong X - Z$, then the homotopy type of the incidence complex $\Delta(D)$ depends only on $\mathcal{I}$ and X. In view of this result, given an arbitrary subscheme $Z \subset X$, the homotopy type of the exceptional divisor after taking an embedded resolution is independent of the choice of the resolution, and is an invariant the pair (X, Z).

If $D = \sum D_i$ is a simple normal crossings (SNC) divisor on a compact Kähler manifold X with each $D_{i_0 \ldots i_p} := D_{i_0} \cap \ldots \cap D_{i_p}$ irreducible, then let $\Delta(D)$ denote the associated dual simplicial complex. Also let $\tilde{\Omega}^r_D$ denote the sheaf of reduced holomorphic r-forms on D, i.e.

$$\tilde{\Omega}^r_D = \{(\alpha_i) \in \bigoplus_i \Omega^r_{D_i} : \alpha_{i|D_{ij}} = \alpha_{j|D_{ij}}, \forall i < j\}$$

and we have that $\tilde{\Omega}^0_D \cong \mathcal{O}_D$. These forms are called reduced because they are precisely the Kähler r-forms on D modulo torsion, i.e. modulo forms supported on the singular locus of D. Now let $\mathcal{H}^q_{dR}(\mathbb{C})$, $\mathcal{H}^q(\Omega^r)$, and $\mathcal{H}^q(\mathcal{O})$ denote the presheaves on Δ which assign to each $\Delta_{i_0 \ldots i_p}$ the vector space $H^q_{dR}(D_{i_0 \ldots i_p}, \mathbb{C})$ (deRham cohomology), $H^q(D_{i_0 \ldots i_p}, \Omega^r_{D_{i_0 \ldots i_p}})$, and $H^q(D_{i_0 \ldots i_p}, \mathcal{O}_{D_{i_0 \ldots i_p}})$ respectively.

Theorem 1.1. *Suppose X is a Kähler manifold (e.g. a smooth projective variety $/\mathbb{C}$) and let $D = \sum D_i$ be a reduced SNC divisor on X with each $D_{i_0} \cap \ldots \cap D_{i_p}$ irreducible, and denote by $\Delta = \Delta(D)$ the dual simplicial complex. Then we have the following isomorphisms:*

$$H^i_{dR}(D, \mathbb{C}) \cong \bigoplus_{p+q=i} H^p(\Delta, \mathcal{H}^q_{dR}(\mathbb{C}))$$
$$H^i(D, \tilde{\Omega}^r_D) \cong \bigoplus_{p+q=i} H^p(\Delta, \mathcal{H}^q(\Omega^r)).$$

In particular, for $r = 0$, we have

$$H^i(D, \mathcal{O}_D) \cong \bigoplus_{p+q=i} H^p(\Delta, \mathcal{H}^q(\mathcal{O})).$$

This formula shows that certain Zariski or deRham cohomology data on an SNC divisor D can be constructed from (1) the dual simplicial incidence complex $\Delta(D)$, and (2) the corresponding cohomology data of the prime components D_i and on their various intersections $D_{i_0 \ldots i_p}$. The recipe is roughly to treat (2) as a presheaf on (1), take simplicial cohomology of $\Delta(D)$ with coefficients in that presheaf, then piece it together. Deligne was the first [4] to establish such a formula for the deRham cohomology of SNC divisors while developing his theory of mixed Hodge structures, and his proof uses the

yoga of weights to establish the second page degeneration of the spectral sequence used in Section 4. In [5], there is a different proof of this degeneration using what is commonly referred to as the $\partial\overline{\partial}$-lemma. Friedman gave a formula of this kind [7] for reduced Kähler differentials, and Stepanov gave one for the structure sheaf in [17]. However, no concrete description of the components of each decomposition was given in these papers. We unify these cohomological formulas in a combinatorial spirit, understanding the pieces of these decompositions as simplicial cohomologies with coefficients in a presheaf of vector spaces. When that presheaf is $\mathcal{H}^0(\mathcal{O}) \cong \mathbb{C}$, we see that the purely combinatorial cohomology $H^i(\Delta, \mathbb{C})$ lives inside $H^i(D, \mathcal{O}_D)$.

It is worth remarking that this formula does not in any way depend on D being an SNC *divisor* with irreducible component intersections. In fact, we may take D to be a suitable SNC cycle, i.e. a sum $D = \sum D_i$ where each component is a smooth irreducible k_i-dimensional subvariety of our ambient n-fold X, and all intersections $D_{i_0 \dots i_p} := D_{i_0} \cap \dots \cap D_{i_p}$ are transversal, smooth, and irreducible of expected dimension $k_{i_0} + \dots + k_{i_p} - np$. Geometrically speaking, an SNC cycle on an n-fold locally looks like a union of coordinate k_i-planes in n-space, and when each $k_i = n-1$, we recover the definition of an SNC divisor.

Corollary 1.2. *In the above setting, we have a Hodge decomposition for D:*

$$H^i_{dR}(D, \mathbb{C}) \cong \bigoplus_{p+q=i} H^p(D, \tilde{\Omega}^q_D).$$

In the toric setting, the cohomology of the structure sheaf of an SNC divisor is purely combinatorial.

Corollary 1.3. *Let X be a smooth complex projective toric variety, and let D be a torus-invariant SNC divisor on X. Then for every $i \geq 0$ we have an isomorphism*

$$H^i(D, \mathcal{O}_D) \cong H^i(\Delta, \mathbb{C}).$$

Inspired by the ideas of [17], we generalize and further develop the theory from the point of view of Stepanov, but note that similar avenues of thought have been explored by G. L. Gordon in the context of monodromy of complex analytic families [9], by P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan in the context of the cohomology of Kähler manifolds [5], by R. Friedman in the context of deformations and smoothings of varieties with normal crossings [7], and by P. Deligne [4], F. El Zein [6], J. Carlson [3], and J. H. M. Steenbrink [15] in the context of mixed Hodge structures. In Section 2, we briefly recall the notion of a presheaf on a simplicial complex and the cohomology of a simplicial complex with coefficients in a presheaf. In Section 3, we establish some exact sequences that become our starting point in building a spectral sequence that we use in Section 4. In Section 5 we state our main results and conjecture the existence of such a formula for effective non-reduced divisors with SNC support. We then show that this would imply the vanishing of $H^i(\Delta(D), \mathbb{C})$ for $i > 0$ when D is the exceptional divisor of a resolution of an isolated rational singularity, and conclude with some examples.

For the purposes of this paper, we shall always assume our divisor $D = \sum r_i D_i$ is such that each intersection $D_{i_0} \cap ... \cap D_{i_p}$ is irreducible. By variety we mean an irreducible one, and we work over $\mathbb{C}$.

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2. Simplicial complexes and cohomology

Let Δ be a finite simplicial complex, and fix a field k.

Definition 2.1. A (covariant) *presheaf* $\mathcal{V}$ of k-vector spaces on Δ consists of the following data:
(a) for every subsimplex Δ' of Δ, a k-vector space $\mathcal{V}(\Delta')$, and
(b) for every inclusion $\Delta'' \subset \Delta'$ of subsimplices of Δ, a "restriction" morphism of k-vector spaces $r = r_{\Delta''\Delta'} : \mathcal{V}(\Delta'') \to \mathcal{V}(\Delta')$,

subject to the usual conditions

(0) $\mathcal{V}(\varnothing) = 0$,
(1) $r_{\Delta'\Delta'}$ is the identity map on $\mathcal{V}(\Delta')$,
(2) if $\Delta''' \subset \Delta'' \subset \Delta'$ are three subsimplices of Δ, then $r_{\Delta'''\Delta'} = r_{\Delta''\Delta'} \circ r_{\Delta'''\Delta''}$.

Thus our presheaf $\mathcal{V}$ on Δ is a covariant functor from the category whose objects are subsimplices of Δ and whose morphisms are inclusions to the category of k-vector spaces. This is slightly different from the usual definition of a presheaf on a topological space, which is usually contravariant.

Definition 2.2. We define the simplicial cohomology of Δ with coefficients in the presheaf $\mathcal{V}$ using the Čech complex.

For each $p \geq 0$, define $C^p = C^p(\Delta, \mathcal{V}) = \bigoplus_{i_0<...<i_p} \mathcal{V}(\Delta_{i_0...i_p})$ where the direct sum is taken over all p-subsimplices $\Delta_{i_0...i_p}$ of Δ. For $v = (v_{i_0...i_p}) \in C^p$, the combinatorial Čech differential $\delta : C^p \to C^{p+1}$ is given by $(\delta v)_{i_0...i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k v_{i_0...\widehat{i_k}...i_{p+1}|\mathcal{V}(\Delta_{i_0...i_{p+1}})} \in \mathcal{V}(\Delta_{i_0...i_{p+1}})$, where the restriction map is $r_{\Delta_{i_0...\widehat{i_k}...i_{p+1}}\Delta_{i_0...i_{p+1}}}$. The compatibility of the restriction morphisms implies that δ is a well-defined k-linear map, and one checks that $\delta^2 = 0$.

We may now define the p^{th} cohomology vector space of Δ with coefficients in $\mathcal{V}$ to be $H^p(\Delta, \mathcal{V}) = h^p(C^{\cdot}(\Delta, \mathcal{V}))$.

2.1. **The Dual Complex of a Simple Normal Crossings Divisor.** Let $D = \sum D_i$ be a simple normal crossings (SNC) divisor on a complex projective variety X. We define the dual complex $\Delta(D)$ to be the CW-complex whose cells are standard simplices $\Delta^k_{i_0 \dots i_p}$ corresponding to the irreducible components $D^k_{i_0 \dots i_p}$ of the nonempty intersections $D_{i_0} \cap \dots \cap D_{i_p} = \bigcup_k D^k_{i_0 \dots i_p}$. That is to say, the 0-simplices (vertices) Δ_i of $\Delta(D)$ correspond to the prime components D_i of D, the 1-simplices (edges) Δ^k_{ij} correspond to the irreducible components D^k_{ij} of all nonempty intersections $D_i \cap D_j = \bigcup_k D^k_{ij}$, the 2-simplices (triangular faces) correspond to the irreducible components of nonempty triple intersections $D_i \cap D_j \cap D_k$, and so on. The $p-1$ simplex $\Delta^j_{i_0 \dots \widehat{i_s} \dots i_p}$ is a face of the p-simplex $\Delta^k_{i_0 \dots i_p}$ iff $D^k_{i_0 \dots i_p} \cap D^j_{i_0 \dots \widehat{i_s} \dots i_p} \neq \varnothing$.

This complex was first introduced by G. L. Gordon to study monodromy in analytic families [9], and whose homotopy type has been more recently studied by D. A. Stepanov in the setting where D is the exceptional divisor of a resolution of an isolated singularity [16], [17], [18], and by A. Thuillier in a more general setting [19].

Notice that if dim $X = n$, then dim $\Delta(D) \leq n-1$, and if $D_{(r)} = \sum r_i D_i$ is an effective non-reduced divisor with SNC support, then $\Delta(D) = \Delta(D_{(r)})$, so there is no confusion in calling this complex Δ. We also restrict our attention to the case when $\Delta(D)$ is a simplicial complex, which happens iff each $D_{i_0 \dots i_p} := D_{i_0} \cap \dots \cap D_{i_p}$ is irreducible.

We consider certain presheaves of $\mathbb{C}$-vector spaces over $\Delta(D)$.

Example 2.3. The constant presheaf $\mathbb{C}$ which assigns to $\mathbb{C}$ to each $\Delta_{i_0 \dots i_p}$, with identity restrictions.

Example 2.4. The presheaf $\mathcal{H}^q(\mathcal{O}_{(r)})$ which assigns $H^q(D^{(r)}_{i_0 \dots i_p}, \mathcal{O}_{D^{(r)}_{i_0 \dots i_p}})$ to each $\Delta_{i_0 \dots i_p}$, where $D^{(r)}_{i_0 \dots i_p}$ is the scheme-theoretic intersection $r_{i_0} D_{i_0} \cap \dots \cap r_{i_p} D_{i_p}$ defined in X by the $\mathcal{O}_X$-ideal $\mathcal{I}^{r_{i_0}}_{i_0} + \dots + \mathcal{I}^{r_{i_p}}_{i_p}$, with the natural $\mathbb{C}$-linear maps on the q^{th} cohomology level induced by the inclusion $D^{(r)}_{i_0 \dots i_p} \hookrightarrow D^{(r)}_{i_0 \dots \widehat{i_s} \dots i_p}$. Here $\mathcal{I}_i$ is the $\mathcal{O}_X$-ideal defining D_i in X. Note that if all $r_i = 1$ then we are in the SNC case and the scheme-theoretic intersection $D_{i_0 \dots i_p}$ coincides with the set-theoretic intersection, in which case we denote $\mathcal{H}^q(\mathcal{O}_{(1)})$ by $\mathcal{H}^q(\mathcal{O})$. Observe that $\mathcal{H}^0(\mathcal{O}) \cong \mathbb{C}$ is the constant presheaf since we have natural isomorphisms $H^0(D_{i_0 \dots i_p}, \mathcal{O}_{D_{i_0 \dots i_p}}) \cong \mathbb{C}$.

Example 2.5. The presheaf $\mathcal{H}^q(\Omega^r)$ which assigns $H^q(D_{i_0 \dots i_p}, \Omega^r_{D_{i_0 \dots i_p}})$ to each $\Delta_{i_0 \dots i_p}$, with the natural $\mathbb{C}$-linear maps on the q^{th} cohomology level induced by the inclusion $D_{i_0 \dots i_p} \hookrightarrow D_{i_0 \dots \widehat{i_s} \dots i_p}$.

Example 2.6. The presheaf $\mathcal{H}^q_{dR}(\mathbb{C})$ which assigns the deRham (or singular) cohomology $H^q_{dR}(D_{i_0 \dots i_p}, \mathbb{C})$ to each $\Delta_{i_0 \dots i_p}$, with the natural $\mathbb{C}$-linear maps on the q^{th} cohomology level induced by the inclusion $D_{i_0 \dots i_p} \hookrightarrow D_{i_0 \dots \widehat{i_s} \dots i_p}$

Thus, using the definitions above, it makes sense to refer to $H^p(\Delta(D), \mathcal{V})$ where $\mathcal{V}$ is any of the above presheaves.

3. Some Technical Lemmas

We first need to establish the exactness of a sheafified Čech complex resolving the structure sheaf of an effective divisor with simple normal crossing support.

Let $D_{(r)} = \sum_{i=1}^{N} r_i D_i$ be an effective divisor with SNC support on a smooth projective variety X (for simplicity we say D is NC), and as before let $D^{(r)}_{i_0 \dots i_p} := r_{i_0} D_{i_0} \cap \dots \cap r_{i_p} D_{i_p}$ be the scheme-theoretic intersection in X. That is to say, if $\mathcal{I}_i \subset \mathcal{O}_X$ is the ideal sheaf defining D_i in X, then $D^{(r)}_{i_0 \dots i_p}$ is defined by the ideal $\mathcal{I}_{i_0}^{r_{i_0}} + \dots + \mathcal{I}_{i_p}^{r_{i_p}}$ in $\mathcal{O}_X$. Notice that $D^{(r)}_i = r_i D_i$. Let $\mathcal{C}^p = \bigoplus_{i_0 < \dots < i_p} \iota_* \mathcal{O}_{D^{(r)}_{i_0 \dots i_p}}$ where $\iota = \iota^{(r)}_{i_0 \dots i_p} : D^{(r)}_{i_0 \dots i_p} \hookrightarrow D_{(r)}$ is the natural inclusion. We have a combinatorial Čech differential $\delta^p : \mathcal{C}^p \to \mathcal{C}^{p+1}$ defined as follows:

given $\alpha = (f_{i_0 \dots i_p}) \in \mathcal{C}^p(U) = \bigoplus_{i_0 < \dots < i_p} \mathcal{O}_{D^{(r)}_{i_0 \dots i_p}}(U \cap D^{(r)}_{i_0 \dots i_p})$ a section of the sheaf $\mathcal{C}^p$ over an open set $U \subset D_{(r)}$, then

$$(\delta\alpha)_{i_0 \dots i_{p+1}}(U) = \sum_{j=0}^{p+1} (-1)^j (f_{i_0 \dots \widehat{i_j} \dots i_{p+1}})_{|U \cap D^{(r)}_{i_0 \dots i_{p+1}}}$$

so $(\delta\alpha)(U) \in \mathcal{C}^{p+1}(U)$, and one easily checks that $\delta^{p+1} \circ \delta^p = 0$ for any $p \geq 0$.. Also, there is a natural injection $\rho : \mathcal{O}_{D_{(r)}} \hookrightarrow \mathcal{C}^0 = \bigoplus_i \iota_* \mathcal{O}_{r_i D_i}$ given by restriction. It is injective because if we are given a section $f \in \mathcal{O}_{D_{(r)}}(U)$ over an affine open subset $U \subset D_{(r)}$, then $\rho_U(f) = 0$ implies that $f_{|U \cap r_i D_i} = 0$ for every i, hence f is in the ideal $I_i^{r_i} \subset \mathcal{O}_{D_{(r)}}(U)$ defining $r_i D_i$ in $D_{(r)}$ over U for every i. But then $f \in \bigcap I_i^{r_i}$ which means that f must vanish on all of $D_{(r)} \cap U = \bigcup r_i D_i \cap U$, i.e. that $f = 0$ on $D_{(r)}$ over U.

Lemma 3.1. *The complex described above*

$$0 \to \mathcal{O}_{D_{(r)}} \to \mathcal{C}^0 \xrightarrow{\delta^0} \mathcal{C}^1 \xrightarrow{\delta^1} \mathcal{C}^2 \xrightarrow{\delta^2} \dots$$

is an exact sequence of $\mathcal{O}_{D_{(r)}}$-modules.

Proof. We have already shown that the first map is injective. We first treat the $p = 0$ joint. Suppose we are given a closed 0-cycle $\alpha = (f_i)$, i.e. $f_j - f_i = 0$ on $D^{(r)}_{ij}$ for every $i < j$. Replacing $D_{(r)}$ by an affine subset of $D_{(r)}$, we have $f_2 - f_1 \in I^{(r)}_{12}$ the ideal of $D^{(r)}_{12}$ in $\mathcal{O}_{D_{(r)}}$. Since $D_{(r)}$ is NC, $I^{(r)}_{12} = I_1^{r_1} + I_2^{r_2}$ where I_i is the ideal of D_i in $\mathcal{O}_{D_{(r)}}$. Hence we can write $f_2 - f_1 = a_2 - a_1$ where $a_i \in I_i^{r_i}$. Now set $f^{(2)} = f_2 - a_2 = f_1 - a_1 \in \mathcal{O}_{D_{(r)}}$, which lifts both f_1 and f_2. By assumption, we have $f^{(2)} - f_3 = 0$ on $D^{(r)}_{13}$ and $D^{(r)}_{23}$, which means

$$f^{(2)} - f_3 \in I^{(r)}_{13} \cap I^{(r)}_{23} = (I_1^{r_1} + I_3^{r_3}) \cap (I_2^{r_2} + I_3^{r_3}) = (I_1^{r_1} \cap I_2^{r_2}) + I_3^{r_3}.$$

All of these ideal equalities hold because $D_{(r)}$ is NC. Write $f^{(2)} - f_3 = a_{12} - a_3$ where $a_{12} \in I_{12}^{(r)}$ and $a_3 \in I_3^{r_3}$, and set $f^{(3)} = f^{(2)} - a_{12} = f_3 - a_3 \in \mathcal{O}_{D_{(r)}}$, which is a lift of f_1, f_2, and f_3. Again by assumption we have $f^{(3)} - f_4 = 0$ on $D_{14}^{(r)}, D_{24}^{(r)}, D_{34}^{(r)}$, hence

$$f^{(3)} - f_4 \in I_{14}^{(r)} \cap I_{24}^{(r)} \cap I_{34}^{(r)} = (I_1^{r_1} + I_4^{r_4}) \cap (I_2^{r_2} + I_4^{r_4}) \cap (I_3^{r_3} + I_4^{r_4}) = (I_1^{r_1} \cap I_2^{r_2} \cap I_3^{r_3}) + I_4^{r_4}$$

Again these ideal equalities hold because $D_{(r)}$ is NC. Hence we may write $f^{(3)} - f_4 = a_{123} - a_4$ where $a_{123} \in I_{123}^{(r)}$ and $a_4 \in I_4^{r_4}$. Set $f^{(4)} = f^{(3)} - a_{123} = f_4 - a_4 \in \mathcal{O}_{D_{(r)}}$, which is a lift of $f_1, ..., f_4$. Continuing this way, we arrive at $f = f^{(N)} \in \mathcal{O}_{D_{(r)}}$ lifting each f_i, and hence mapping to our cycle (f_i).

For the general $p \geq 1$ case, we induct on the dimension of the ambient variety X. If dim $X = 0$, then there is nothing to show. To show the inductive step, we induct on N, the number of components of $D_{(r)}$, the case $N = 1$ being trivial, since our complex is simply

$$0 \to \mathcal{O}_{D_{(r)}} \to \mathcal{O}_{D_{(r)}} \to 0$$

which is obviously exact. Let $\alpha = (f_{i_0 \dots i_p}) \in \mathcal{C}^p$ be a closed p-cycle, and write

$$\alpha = \alpha_{\neq 1} \oplus \alpha_1 = (f_{i_0 \dots i_p})_{i_0 > 1} \oplus (f_{1 i_1 \dots i_p})$$

The Čech differential of the complex associated to $r_2 D_2 + ... + r_N D_N$ acts the same way as does the Čech differential for $D_{(r)}$ on the $\alpha_{\neq 1}$ component, hence the induction hypothesis on N gives us a $\beta_{\neq 1} = (g_{i_0 \dots i_{p-1}})_{i_0 > 1} \in \bigoplus_{1 < i_0 < \dots < i_{p-1}} \iota_* \mathcal{O}_{D^{(r)}_{i_0 \dots i_{p-1}}}$ such that $(\delta^{p-1} \beta_{\neq 1})_{i_0 \dots i_p} = (\alpha_{\neq 1})_{i_0 \dots i_p}$, $i_0 > 1$.

Now set $D_i' = D_{1i}^{(r)} = r_1 D_1 \cap r_i D_i$ for $i = 2, ..., N$ so that $D' = \sum_{i=2}^{N} D_i'$ is an effective divisor on D_1 with SNC support. Denote by $D'_{i_1 \dots i_p} = D'_{i_1} \cap \dots \cap D'_{i_p}$ the scheme-theoretic intersection in D_1, and consider

$$(g_{i_1 \dots i_p | D^{(r)}_{1 i_1 \dots i_p}} - f_{1 i_1 \dots i_p}) \in \bigoplus_{1 < i_1 < \dots < i_p} \mathcal{O}_{D'_{i_1 \dots i_p}} = \bigoplus_{1 < i_1 < \dots < i_p} \mathcal{O}_{D^{(r)}_{1 i_i \dots i_p}}$$

as a $p-1$ cocycle in the complex associated to D'. This cocycle is closed, i.e. we have that

$$(g_{i_2 \dots i_{p+1}} - f_{1 i_2 \dots i_{p+1}})_{|D^{(r)}_{1 i_1 \dots i_{p+1}}} - (g_{i_1 i_3 \dots i_{p+1}} - f_{1 i_1 i_3 \dots i_{p+1}})_{|D^{(r)}_{1 i_1 \dots i_{p+1}}} + \dots + (-1)^p (g_{i_1 \dots i_p} - f_{1 i_1 \dots i_p})_{|D^{(r)}_{1 i_1 \dots i_{p+1}}}$$

$$= \sum_{j=1}^{p+1} (-1)^{j-1} g_{i_1 \dots \widehat{i_j} \dots i_{p+1} | D^{(r)}_{1 i_1 \dots i_{p+1}}} + \sum_{j=1}^{p+1} (-1)^j f_{1 i_1 \dots \widehat{i_j} \dots i_{p+1} | D^{(r)}_{1 i_1 \dots i_{p+1}}}$$

$$= f_{i_1 \dots i_{p+1} | D^{(r)}_{1 i_1 \dots i_{p+1}}} + \sum_{j=1}^{p+1} (-1)^j f_{1 i_1 \dots \widehat{i_j} \dots i_{p+1} | D^{(r)}_{1 i_1 \dots i_{p+1}}} = 0$$

by our assumption that α is a closed p-cycle and by our construction of the $g_{i_1...i_p}$. Thus, by our induction hypothesis on dim $D_1 <$ dim X, this cocycle is a coboundary, meaning that there exists a $p-2$-cycle $(h_{i_1...i_{p-1}}) =: (g_{1i_1...i_{p-1}}) =: \beta_1 \in \bigoplus_{1<i_1<...<i_{p-1}} \mathcal{O}_{D'_{i_1...i_{p-1}}} = \bigoplus_{1<i_1<...<i_{p-1}} \mathcal{O}_{D^{(r)}_{1i_1...i_{p-1}}}$ such that for every $1 < i_1 < ... < i_p$, we have that

$$(g_{1i_2...i_p} - g_{1i_1i_3...i_p} + ... + (-1)^{p-1} g_{1i_1...i_{p-1}})_{|D^{(r)}_{1i_1...i_p}} = g_{i_1...i_p|D^{(r)}_{1i_1...i_p}} - f_{1i_1...i_p}.$$

Hence we may take $\beta = \beta_{\neq 1} \oplus \beta_1 = (g_{i_0...i_{p-1}})_{i_0>1} \oplus (g_{1i_1...i_{p-1}})$ and one sees that by construction we have $\delta^{p-1}(\beta) = \alpha$ for every $i_0 < ... < i_p$, which completes the proof. ☐

We show briefly how the usual Kähler forms on a (reduced) SNC divisor $D = \sum D_i$ on X are related to reduced forms, following Friedman [7] Section 1. Since D is reduced and a local complete intersection, the conormal sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X|D} \to \Omega^1_D \to 0$$

is exact, and allows us to define the Kähler differentials on D, taking $\Omega^r_D := \wedge^r \Omega^1_D$. There is a natural map $\Omega^r_D \to \bigoplus \iota_* \Omega^r_{D_i}$ whose kernel τ^r_D consists of those differentials supported on Sing(D). We may then define $\tilde{\Omega}^r_D := \Omega^r_D/\tau^r_D$. One can directly show that this agrees with the definition given in the introduction.

Lemma 3.2. *Let $D = \sum D_i$ be a SNC divisor, and let $\mathcal{E}^p_r = \bigoplus_{i_0<...<i_p} \iota_* \Omega^r_{D_{i_0...i_p}}$. Then we have an exact sequence of $\mathcal{O}_D$-modules*

$$0 \to \tilde{\Omega}^r_D \to \mathcal{E}^0_r \xrightarrow{\delta^0} \mathcal{E}^1_r \xrightarrow{\delta^1} \mathcal{E}^2_r \xrightarrow{\delta^2} ...$$

where the first map is inclusion and the other maps are the Čech differentials as in Lemma 3.1.

Proof. R. Friedman gives a proof of this lemma in [7] Proposition 1.5 using a double induction scheme similar to ours in Lemma 3.1. We sketch an alternate proof: since $\tilde{\Omega}^r_D$ is defined to be $\ker \delta^0$ we get exactness at the $p = 0$ joint for free. For the remaining joints, note that this is a local question and choose local coordinates of $D_{i_0...i_{p+1}}$ such that we may extend them to local coordinates of $D_{i_0...i_p}$ and $D_{i_0...i_{p-1}}$. Then noting that the various wedges $dz_{j_1} \wedge ... \wedge dz_{j_r}$ form a local basis of $\Omega^r_{D_{i_0...i_p}}$, a form $(\alpha) \in \mathcal{E}^p_r$ being δ-closed implies the corresponding cocycles of holomorphic coefficients of the basis elements are each δ-closed as well. We can lift these cocycles to coboundaries by Lemma 3.1, thus giving a natural choice of a coboundary form $(\beta) \in \mathcal{E}^{p-1}_r$ which δ sends to (α). ☐

Remark 3.3. The above lemmas generalize in a straightforward way to give such exact sequences for an SNC *cycle* on X.

4. The Spectral Sequence

We first work out in detail a spectral sequence associated to a SNC divisor $D = \sum D_i$, and later indicate the adjustments needed in the more general case where $D_{(r)} = \sum r_i D_i$ is an effective divisor with SNC support. This spectral sequence for the SNC case (or one similar to it) has been worked out in [4], [5], [10], and [14] for deRham cohomology, in [17] for the structure sheaf $\mathcal{O}_D$, and is mentioned in [7] for reduced forms $\tilde{\Omega}^r_D$. We follow Stepanov's presentation in [17], but work with the sheaf of reduced holomorphic r-forms, which for $r = 0$ includes the structure sheaf. As usual we assume that each intersection $D_{i_0 \dots i_p}$ is irreducible. We have by Lemma 3.2 that

$$0 \to \tilde{\Omega}^r_D \to \mathcal{E}^0_r \xrightarrow{\delta^0} \mathcal{E}^1_r \xrightarrow{\delta^1} \mathcal{E}^2_r \xrightarrow{\delta^2} \dots$$

is exact, where $\mathcal{E}^p_r = \bigoplus_{i_0 < \dots < i_p} \iota_* \Omega^r_{D_{i_0 \dots i_p}}$. This implies that the complexes

$$\Omega^* : \tilde{\Omega}^r_D \to 0 \to 0 \to \dots$$

and

$$\mathcal{E}^*_r : \mathcal{E}^0_r \to \mathcal{E}^1_r \to \mathcal{E}^2_r \to \dots$$

are quasiisomorphic. Clearly the hypercohomologies of Ω^* are just the cohomologies of $\tilde{\Omega}^r_D$, i.e. $\mathbb{H}^i(\Omega^*) \cong H^i(D, \tilde{\Omega}^r_D)$. To calculate the hypercohomologies of $\mathcal{E}^*_r$, we take Dolbeault resolutions

$$0 \to \mathcal{E}^p_r \to \mathcal{E}^{p,0}_r \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}_r \xrightarrow{\overline{\partial}} \mathcal{E}^{p,2}_r \xrightarrow{\overline{\partial}} \dots$$

where $\mathcal{E}^{p,q}_r = \bigoplus_{i_0 < \dots < i_p} \iota_* \mathcal{A}^{r,q}_{D_{i_0 \dots i_p}}$, $\mathcal{A}^{r,q}_{D_{i_0 \dots i_p}}$ is the sheaf of differential forms of type (r, q) on $D_{i_0 \dots i_p}$ and the maps are the Dolbeault differentials $\overline{\partial}$.

Now consider the bigraded collection of $\mathbb{C}$-vector spaces

$$C^{p,q} := H^0(D, \mathcal{E}^{p,q}_r) = \bigoplus_{i_0 < \dots < i_p} H^0(D_{i_0 \dots i_p}, \mathcal{A}^{r,q}_{D_{i_0 \dots i_p}}) = \bigoplus_{i_0 < \dots < i_p} A^{r,q}_{D_{i_0 \dots i_p}}$$

equipped with the Dolbeault differential $\overline{\partial} : C^{p,q} \to C^{p,q+1}$ and the combinatorial Čech differential $\delta : C^{p,q} \to C^{p+1,q}$ defined as follows: given $\alpha = (\alpha_{i_0 \dots i_p}) \in C^{p,q}$, we set

$$(\delta\alpha)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^{q+j} (\alpha_{i_0 \dots \hat{i_j} \dots i_{p+1}})_{|D_{i_0 \cdots i_{p+1}}}.$$

We have $\overline{\partial}\delta + \delta\overline{\partial} = 0$, so that $(C^{*,*}, \delta, \overline{\partial})$ is a bicomplex. Take (C^*, d) to be the total complex, i.e. $C^m = \bigoplus_{p+q=m} C^{p,q}$ and $d = \delta + \overline{\partial}$. We may now express the hypercohomologies of $\mathcal{E}^*_r$ as the cohomologies of (C^*, d), that is to say $\mathbb{H}^i(\mathcal{E}^*_r) \cong h^i(C^*, d)$.

The filtration $F^pC^m = \bigoplus_{\substack{p'+q=m \\ p'\geq p}} C^{p',q}$ on the complex (C^*, d) gives rise to a spectral sequence E_* that converges to the cohomologies $h^*(C^*, d)$ (see [11] Ch.3, section 5) and

$$E_0^{p,q} \cong C^{p,q};$$

$$E_1^{p,q} \cong H^q_{\overline{\partial}}(C^{p,*});$$

$$E_2^{p,q} \cong H^p_\delta(H^q_{\overline{\partial}}(C^{*,*})).$$

We have that

$$E_1^{p,q} \cong \bigoplus_{i_0<\ldots<i_p} H^q_{\overline{\partial}}(A^{r,*}_{D_{i_0\ldots i_p}}) \cong \bigoplus_{i_0<\ldots<i_p} H^q(D_{i_0\ldots i_p}, \Omega^r_{D_{i_0\ldots i_p}}).$$

Observe that the cochain complex

$$0 \to E_1^{0,q} \to E_1^{1,q} \to E_1^{2,q} \to \ldots$$

with the induced Čech differentials δ is isomorphic to the cochain complex

$$0 \to \bigoplus_i H^q(D_i, \Omega^r_{D_i}) \to \bigoplus_{i_0<i_1} H^q(D_{i_0i_1}, \Omega^r_{D_{i_0i_1}}) \to \bigoplus_{i_0<i_1<i_2} H^q(D_{i_0i_1i_2}, \Omega^r_{D_{i_0i_1i_2}}) \to \ldots$$

that one uses to calculate the cohomologies of $\Delta(D)$ with coefficients in the presheaf $\mathcal{H}^q(\Omega^r)$. Hence

$$E_2^{p,q} \cong H^p_\delta(E_1^{*,q}) \cong H^p(\Delta(D), \mathcal{H}^q(\Omega^r)).$$

We now show how to get such a spectral sequence for the structure sheaf of an effective non-reduced divisor $D_{(r)} = \sum r_i D_i$ with SNC support. Since we have our exact sequence from Lemma 3.1 in this setting, the only missing ingredient is a Dolbeault type resolution of the non-reduced structure sheaves $\mathcal{O}_{D^{(r)}_{i_0\ldots i_p}}$, where $D^{(r)}_{i_0\ldots i_p} = r_{i_0}D_{i_0} \cap \ldots \cap r_{i_p}D_{i_p}$ is the scheme-theoretic intersection defined by $\mathcal{I} := \mathcal{I}^{r_{i_0}}_{i_0} + \ldots + \mathcal{I}^{r_{i_p}}_{i_p} \subset \mathcal{O}_X$. If we have this, we may then construct the bicomplex $(C^{*,*}, \delta, \overline{\partial})$ to obtain the spectral sequence as in the SNC case above. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[d] & \mathcal{I}\mathcal{A}_X^{0,0} \arrow[r, "\overline{\partial}"] \arrow[d] & \mathcal{I}\mathcal{A}_X^{0,1} \arrow[r, "\overline{\partial}"] \arrow[d] & \ldots \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{A}_X^{0,0} \arrow[r, "\overline{\partial}"] \arrow[d] & \mathcal{A}_X^{0,1} \arrow[r, "\overline{\partial}"] \arrow[d] & \ldots \\
0 \arrow[r] & \mathcal{O}_X/\mathcal{I} \arrow[r] & \mathcal{A}_X^{0,0}/\mathcal{I}\mathcal{A}_X^{0,0} \arrow[r, "\overline{\partial}"] & \mathcal{A}_X^{0,1}/\mathcal{I}\mathcal{A}_X^{0,1} \arrow[r, "\overline{\partial}"] & \ldots
\end{tikzcd}$$

The diagram commutes because our ideal is locally generated by holomorphic functions, which commute with $\overline{\partial}$, and it is easy to see that $\overline{\partial}(\mathcal{I}\mathcal{A}_X^{0,q}) \subset \mathcal{I}\mathcal{A}_X^{0,q+1}$. We would like to show that the third row is exact, which would give us the acyclic resolution that we need. Indeed, each $\mathcal{A}_X^{0,q}/\mathcal{I}\mathcal{A}_X^{0,q}$ is a quotient of acyclic sheaves and is hence acyclic for the global section functor. Since the columns are all short exact and the second row is the Dolbeault resolution of $\mathcal{O}_X$ (and is thus exact), it suffices by the Snake Lemma to show that the first row is exact. To do this we simply reprove the Dolbeault lemma while keeping track of the holomorphic coefficients in $\mathcal{I}$.

Lemma 4.1. *Let $\mathcal{I} \subset \mathcal{O}_X$ be a finitely generated ideal sheaf, and let $f = f(z_1, ..., z_n) \in \mathcal{I}\mathcal{A}_X^{0,0}(U)$ be a C^∞ function on an open subset $U \subset X$ which is holomorphic in the variables z_l, $l > q$. Then there exists an open subset $V \subset U$ and a C^∞ function $g \in \mathcal{I}\mathcal{A}_X^{0,0}(V)$, holomorphic in the variables z_l, $l > q$, such that $\frac{\partial g}{\partial \overline{z}_q} = f$.*

Proof. Write $\mathcal{I}(U) = I = (x_1, ..., x_k)$ and $f = \sum x_j f_j$, where the sum runs over the generators of I, and f_j are C^∞. It is well known (see [20] Prop 2.32) that locally we can find C^∞ functions g_j such that $\frac{\partial g_j}{\partial \overline{z}_q} = f_j$, by taking $g_j = \frac{1}{2\pi i} \int \frac{f_j(z_1, ..., w_q, ..., z_n)}{w_q - z_q} dw_q \wedge d\overline{w}_q$. Now set $g = \sum x_j g_j$. Since the x_j are holomorphic, we have that

$$\frac{\partial g}{\partial \overline{z}_q} = \sum x_j \frac{\partial g_j}{\partial \overline{z}_q} = \sum x_j f_j = f.$$

Also, g remains holomorphic in the variables z_l, $l > q$ since we may pass the x_j and the operator $\frac{\partial}{\partial \overline{z}_l}$ through the integral. □

Lemma 4.2. *The first row in the above diagram is exact.*

Proof. The proof is identical to that of the Dolbeault lemma (see [20] Prop 2.31 and Prop 2.36) which uses a lemma similar to the one above, the only addition being that the coefficients of the forms in question are always in $\mathcal{I}\mathcal{A}_X^{0,0}$. □

Let $\mathcal{A}^{0,q}_{D^{(r)}_{i_0\dots i_p}}$ denote $\mathcal{A}^{0,q}_X/\mathcal{I}\mathcal{A}^{0,q}_X$, where $\mathcal{I} := \mathcal{I}^{r_{i_0}}_{i_0} + \dots + \mathcal{I}^{r_{i_p}}_{i_p}$ is the $\mathcal{O}_X$ ideal that defines the scheme-theoretic intersection $D^{(r)}_{i_0\dots i_p} = r_{i_0}D_{i_0} \cap \dots \cap r_{i_p}D_{i_p}$ in X. Also let $\mathcal{C}^{p,q} = \bigoplus_{i_0<\dots i_p} \mathcal{A}^{0,q}_{D^{(r)}_{i_0\dots i_p}}$ and let $C^{p,q} = \bigoplus_{i_0<\dots i_p} A^{0,q}_{D_{i_0\dots i_p}}$ denote the global sections.

Thus, we have our spectral sequence for the structure sheaf of an effective non-reduced divisor with SNC support. We now give a quick Hodge-theoretic proof of the second page degeneration of our spectral sequence in the reduced case, similar to the one given in [5].

Lemma 4.3 ($\partial\overline{\partial}$-lemma)**.** *Suppose ω is a global (r,q) form on a Kähler manifold $D_{i_0\dots i_p}$ that is both ∂ and $\overline{\partial}$ closed. If ω is $\overline{\partial}$ exact or ∂ exact, then $\omega = \partial\overline{\partial}\gamma$, for some form γ of type $(r-1,q-1)$.*

Proof. See either [5] or [20] Prop 6.17. The idea is to use the decomposition of the space of forms into the kernel and image of the Laplacian, and employ the Kähler identities. $\square$

Now let $K^{p,q} = \bigoplus_{i_0<\dots<i_p} \ker\partial \subset \bigoplus_{i_0<\dots<i_p} A^{r,q}_{D_{i_0\dots i_p}} = C^{p,q}$, where $\partial : A^{r,q} \to A^{r+1,q}$ is our usual operator and let $H^{p,q} = \bigoplus_{i_0<\dots<i_p} H^q(D_{i_0\dots i_p}, \Omega^r_{D_{i_0\dots i_p}})$. Consider the natural maps of double complexes

$$(H^{*,*}, \delta, 0) \leftarrow (K^{*,*}, \delta, \overline{\partial}) \hookrightarrow (C^{*,*}, \delta, \overline{\partial})$$

where the first map is via the identification of ∂ and $\overline{\partial}$ cohomology on a Kähler manifold. We claim that the induced maps on the first pages of the associated spectral sequences is an isomorphism. Indeed, we saw above that the first page of the spectral sequence associated to $C^{*,*}$ is just $E_1^{p,q} \cong \bigoplus_{i_0<\dots<i_p} H^q_{\overline{\partial}}(C^{p,*}) = \bigoplus_{i_0<\dots<i_p} H^q(D_{i_0\dots i_p}, \Omega^r_{D_{i_0\dots i_p}})$, so it suffices to show that the natural map

$$\ker\partial \cap \ker\overline{\partial}/\ker\partial \cap \operatorname{im}\overline{\partial} = \ker\partial \cap \ker\overline{\partial}/\operatorname{im}\partial\overline{\partial} \to H^q(D_{i_0\dots i_p}, \Omega^r_{D_{i_0\dots i_p}}) \cong \ker\overline{\partial}/\operatorname{im}\overline{\partial}$$

is an isomorphism for each $i_0 < \dots < i_p$. To see injectivity, given a (r,q) form $\alpha \in \ker\partial \cap \ker\overline{\partial}$, it maps to 0 in cohomology iff α is $\overline{\partial}$ exact. Applying the $\partial\overline{\partial}$-lemma gives $\alpha = \partial\overline{\partial}\gamma$, hence $\alpha = 0$. For surjectivity, given a $\overline{\partial}$ closed form β, we may apply the $\partial\overline{\partial}$-lemma to $\partial\beta$ and conclude that $\partial\beta = \partial\overline{\partial}\gamma$. Then $\beta - \overline{\partial}\gamma$ also represents $[\beta]$ and lies in $\ker\partial \cap \ker\overline{\partial}$, and hence maps to $[\beta]$.

So the induced map on first pages is an isomorphism, hence all higher pages are isomorphic by the Zeeman Comparison Theorem, see [21] Theorem 5.2.12. However, the total differential of $H^{*,*}$ is just δ, so the corresponding spectral sequence must collapse at the second page, and hence so must the one corresponding to $C^{*,*}$.

So our spectral sequence degenerates at the second page, i.e. $E_2 = E_\infty$. Our calculations give

$$H^i(D, \tilde{\Omega}^r_D) \cong \mathbb{H}^i(\Omega^*) \cong \mathbb{H}^i(\mathcal{E}^*_r) \cong h^i(C^*, d) \cong \bigoplus_{p+q=i} E_2^{p,q} \cong \bigoplus_{p+q=i} H^p(\Delta(D), \mathcal{H}^q(\Omega^r)).$$

Taking $r = 0$ we see that

$$H^i(\Delta, \mathbb{C}) \cong H^i(\Delta, \mathcal{H}^0(\mathcal{O})) \cong E_2^{i,0} \cong E_\infty^{i,0} \hookrightarrow H^i(D, \mathcal{O}_D).$$

A very similar argument holds for the second page degeneration of the spectral sequence for the deRham cohomology of a SNC divisor on a Kähler manifold, giving the same kind of decomposition. One shows that the sequence of $\mathcal{O}_D$-modules

$$0 \to \mathbb{C} \to \bigoplus_i \iota_* \mathbb{C} \xrightarrow{\delta^0} \bigoplus_{i_0 < i_1} \iota_* \mathbb{C} \xrightarrow{\delta^1} \bigoplus_{i_0 < i_1 < i_2} \iota_* \mathbb{C} \xrightarrow{\delta^2} \ldots$$

is exact, resolves each $\mathbb{C}$ on $D_{i_0 \ldots i_p}$ by the deRham resolution

$$0 \to \mathbb{C} \to \mathcal{A}^0_{D_{i_0 \ldots i_p}} \xrightarrow{d^0} \mathcal{A}^1_{D_{i_0 \ldots i_p}} \xrightarrow{d^1} \mathcal{A}^2_{D_{i_0 \ldots i_p}} \xrightarrow{d^2} \ldots$$

and sets $C^{p,q} := \bigoplus_{i_0 < \ldots < i_p} H^0(D_{i_0 \ldots i_p}, \mathcal{A}^q_{D_{i_0 \ldots i_p}})$. Then $(C^{*,*}, d, \delta)$ is a bicomplex whose total complex $(C^*, d + \delta)$ has the same cohomology as $H^*_{dR}(D, \mathbb{C})$. The usual filtration on the bicomplex gives a spectral sequence as above, and one argues that this degenerates at the second page either by hand, using the Hodge decomposition and the $\partial\overline{\partial}$-lemma, or more compactly by considering three double complexes as above. In the latter scenario, the middle bicomplex is associated to $\ker d^c$, where $d^c = \sqrt{-1}(\overline{\partial} - \partial)$, and the dd^c-lemma is used, which is equivalent to the $\partial\overline{\partial}$-lemma. For details of either argument, see [14] or [5] respectively.

5. Main Results, Conjectures, and Examples

Applying the spectral sequence discussed in the previous section to the exact sequences from the lemmas in Section 3 yields formulas, which are essentially a reformulation of Deligne's result for deRham cohomology and Stepanov's result for the structure sheaf cohomology of a SNC divsor in the language of cohomologies of presheaves on the associated dual simplicial complex. We summarize them now.

Theorem 5.1. *Suppose X is a Kähler manifold and let $D = \sum D_i$ be a reduced SNC divisor on X with each $D_{i_0} \cap ... \cap D_{i_p}$ irreducible, and denote by $\Delta = \Delta(D)$ the dual simplicial complex. Then we have the following isomorphisms:*

$$H^i_{dR}(D, \mathbb{C}) \cong \bigoplus_{p+q=i} H^p(\Delta, \mathcal{H}^q_{dR}(\mathbb{C}))$$

$$H^i(D, \tilde{\Omega}^r_D) \cong \bigoplus_{p+q=i} H^p(\Delta, \mathcal{H}^q(\Omega^r)).$$

In particular, for $r = 0$, we have

$$H^i(D, \mathcal{O}_D) \cong \bigoplus_{p+q=i} H^p(\Delta, \mathcal{H}^q(\mathcal{O})).$$

These formulas are interesting in their own right because they show that no new cohomological information is created when smooth divisors are added to one another in a normal crossings fashion, provided we know how they intersect via the dual incidence complex $\Delta(D)$. We also recover a Hodge decomposition for a SNC divisor on a Kähler manifold.

Corollary 5.2. *In the setting above, we have an isomorphism:*

$$H^i_{dR}(D, \mathbb{C}) \cong \bigoplus_{r+q=i} H^q(D, \tilde{\Omega}^r_D)$$

Proof. The Hodge decomposition for each $D_{i_0 \dots i_p}$ gives natural isomorphisms

$$H^q_{dR}(D_{i_0 \dots i_p}, \mathbb{C}) \cong \bigoplus_{r+s=q} H^s(D_{i_0 \dots i_p}, \Omega^r_{D_{i_0 \dots i_p}})$$

for every $i_0 < ... < i_p$ so we have isomorphisms $\mathcal{H}^q_{dR}(\mathbb{C}) \cong \bigoplus_{r+s=q} \mathcal{H}^s(\Omega^r)$ as presheaves on Δ. Using Theorem 5.1 above we have a chain of isomorphisms

$$H^i_{dR}(D, \mathbb{C}) \cong \bigoplus_{p+q=i} H^p(\Delta, \mathcal{H}^q_{dR}(\mathbb{C})) \cong \bigoplus_{p+q=i} H^p(\Delta, \bigoplus_{r+s=q} \mathcal{H}^s(\Omega^r)) \cong \bigoplus_{p+r+s=i} H^p(\Delta, \mathcal{H}^s(\Omega^r))$$

$$\cong \bigoplus_{r=0}^{i} \bigoplus_{p+s=i-r} H^p(\Delta, \mathcal{H}^s(\Omega^r)) \cong \bigoplus_{r=0}^{i} H^{i-r}(D, \tilde{\Omega}^r_D) \cong \bigoplus_{r+q=i} H^q(D, \tilde{\Omega}^r_D).$$

□

We make the following conjecture for the more general non-reduced case.

Conjecture 5.3. In the setting of Theorem 5.1, let $D_{(r)} = \sum r_i D_i$ be an effective non-reduced divisor with SNC support on X, assume each $D_{i_0} \cap ... \cap D_{i_p}$ irreducible, and denote by $\Delta = \Delta(D_{(r)})$ the dual simplicial complex. Then the spectral sequence constructed in Section 4 for the structure sheaf $\mathcal{O}_{D_{(r)}}$ degenerates at the second page, yielding the isomorphism

$$H^i(D, \mathcal{O}_{D_{(r)}}) \cong \bigoplus_{p+q=i} H^p(\Delta, \mathcal{H}^q(\mathcal{O}_{(r)})).$$

This seems quite hard, since we do not have the ∂ operator and we cannot do Hodge theory. One encounters the same problem trying to give such a formula for an arbitrary vector bundle on a (reduced) SNC divisor D.

5.1. **The Toric Setting.** In the setting where X is a smooth projective toric variety $/\mathbb{C}$ and D is a torus-invariant SNC divisor, our formula simplifies greatly, and we see that the Zariski cohomology of the structure sheaf $\mathcal{O}_D$ is precisely the simplicial cohomology of $\Delta(D)$.

Corollary 5.4. *In the toric setting above, we have an isomorphism*

$$H^i(D, \mathcal{O}_D) \cong H^i(\Delta(D), \mathbb{C}).$$

Proof. It is well-known that for a globally generated line bundle on a simplicial complete toric variety, the higher cohomology groups vanish (see [8] Section 3.5). Thus the presheafs $\mathcal{H}^q(\mathcal{O})$ are 0 for $q > 0$ since $H^q(D_{i_0 \dots i_p}, \mathcal{O}_{D_{i_0 \dots i_p}}) = 0$ for $q > 0$, and we are done by Theorem 5.1 since we saw that $\mathcal{H}^0(\mathcal{O}) \cong \mathbb{C}$ in Example 2.4. □

Remark 5.5. More generally, on a smooth projective complex variety X (not necessarily toric) suppose that we have an SNC divisor D such that $H^q(D_{i_0 \dots i_p}, \mathcal{O}_{D_{i_0, \dots i_p}}) = 0$ for every $q > 0$ and each $i_0 < ... < i_p$. Then Theorem 5.1 again immediately gives that

$$H^i(D, \mathcal{O}_D) \cong H^i(\Delta(D), \mathbb{C}).$$

Hence invariants such as the Euler characteristic coincide, i.e. $\chi(\mathcal{O}_D) = \chi(\Delta(D))$.

5.2. **Rational Singularities.** Our starting point in this section is the following result of Stepanov, generalized to arbitrary singular loci over a perfect field by A. Thuillier in [19].

Theorem 5.6 (Stepanov)**.** *The homotopy type of the dual complex $\Delta(D)$ of an SNC divisor D associated to a resolution of an isolated singularity $o \in Y$ is independent of the choice of the resolution. That is to say, the homotopy type of $\Delta(D)$ depends only on the isolated singularity o and the ambient variety Y.*

Proof. The proof uses the weak factorization theorem of birational maps to reduce to the case of a blowup. See [16] for details. ☐

Notice that by taking additional blowups, we can always ensure the irreducibility of each intersection $D_{i_0 \dots i_p}$ without affecting the homotopy type of $\Delta(D)$. Such a resolution is called *good*. As we mentioned in the introduction, Stepanov has shown the contractibility of Δ for various kinds of isolated singularities: toric, Brieskorn, 3-dimensional terminal, 3-dimensional rational hypersurface; see [16], [17], and [18] for details. Stepanov asks if $\Delta(D)$ is contractible for all isolated rational singularities of dimension ≥ 3. We conjecture a weaker, but more plausible assertion.

Conjecture 5.7. Suppose $f : X \to Y$ is a good resolution of an isolated rational singularity $o \in Y$ so that the exceptional divisor $D_{(r)} = \sum r_i D_i$ is effective with SNC support. Then $H^i(\Delta, \mathbb{C}) = 0$ for all $i > 0$.

We show that Conjecture 5.3 implies Conjecture 5.7. For any $i_0 < \dots < i_p$ we have a short exact sequence (as coherent sheaves on $D^{(r)}_{i_0 \dots i_p}$)

$$0 \to \mathcal{I}^{(r)}_{i_0 \dots i_p} \to \mathcal{O}_{D^{(r)}_{i_0 \dots i_p}} \to \mathcal{O}_{D_{i_0 \dots i_p}} \to 0$$

where $\mathcal{I}^{(r)}_{i_0 \dots i_p} = \dfrac{(\mathcal{I}_{i_0} + \dots + \mathcal{I}_{i_p})}{(\mathcal{I}^{r_{i_0}}_{i_0} + \dots + \mathcal{I}^{r_{i_p}}_{i_p})}$ is the ideal defining $D_{i_0 \dots i_p}$ in $D^{(r)}_{i_0 \dots i_p}$. Taking global sections we get another short exact sequence

$$0 \to H^0(\mathcal{I}^{(r)}_{i_0 \dots i_p}) \to H^0(\mathcal{O}_{D^{(r)}_{i_0 \dots i_p}}) \to \mathbb{C} \to 0$$

where the last restriction map is surjective because the middle term is a nonzero complex vector space (it contains the constant functions) and the second map is restriction. But a short exact sequence of vector spaces splits, hence we have

$$H^0(\mathcal{O}_{D^{(r)}_{i_0 \dots i_p}}) \cong \mathbb{C} \oplus H^0(\mathcal{I}^{(r)}_{i_0 \dots i_p})$$

Since we get such natural isomorphisms for every $i_0 < \dots < i_p$, we conclude that

$$\mathcal{H}^0(\mathcal{O}_{D_{(r)}}) \cong \mathbb{C} \oplus \mathcal{H}^0(\mathcal{I}^{(r)})$$

as presheaves on Δ, where $\mathcal{H}^0(\mathcal{I}^{(r)})(\Delta_{i_0 \dots i_p}) = H^0(D^{(r)}_{i_0 \dots i_p}, \mathcal{I}^{(r)}_{i_0 \dots i_p})$. It follows that we have an isomorphism on the level of cohomology

$$H^i(\Delta, \mathcal{H}^0(\mathcal{O}_{D_{(r)}})) \cong H^i(\Delta, \mathbb{C}) \oplus H^i(\Delta, \mathcal{H}^0(\mathcal{I}^{(r)}))$$

so that we have inclusions

$$H^i(\Delta, \mathbb{C}) \hookrightarrow H^i(\Delta, \mathcal{H}^0(\mathcal{O}_{(r)})) \hookrightarrow H^i(D_{(r)}, \mathcal{O}_{D_{(r)}}) \cong (R^i f_* \mathcal{O}_X)_o$$

and the last vector space is 0 since our singularity was assumed to be rational.

Remark 5.8. Notice that we do not need the entirety of Conjecture 5.3 to prove Conjecture 5.7 in the argument given above. It suffices to have an inclusion

$$H^i(\Delta, \mathbb{C}) \hookrightarrow H^i(D_{(r)}, \mathcal{O}_{D_{(r)}}) \cong (R^i f_* \mathcal{O}_X)_o$$

of the combinatorial part of the cohomology into the cohomology of the structure sheaf of $D_{(r)}$.

We state the following interesting result of Stepanov.

Theorem 5.9 (Stepanov)**.** *If $\Delta = \Delta(D)$ is the dual complex associated to a good resolution of an isolated hypersurface singularity of dimension ≥ 3, then $\pi_1(\Delta) = 0$, i.e Δ is simply connected.*

Proof. See [17], Theorem 3.1. The idea is to consider the link M of the isolated singularity, which is simply connected, and think of M as the border of a tubular neighborhood of the exceptional divisor D, which gives a surjective map $\varphi : M \to D$ whose fibers are tori. One concludes that $\pi_1(D) = 0$ and then argues that this implies that $\pi_1(\Delta(D)) = 0$. □

Unfortunately, even if Conjecture 5.7 were true, we still would not know the cohomology of Δ with coefficients in $\mathbb{Z}$ so we could not conclude that Δ is contractible in the case of an isolated rational hypersurface singularity of dimension ≥ 3. But, we could say this of the rationalization of Δ. Roughly speaking, the rationalization of a simply connected CW complex Δ is a canonically constructed simply connected CW complex Δ_0, unique up to homotopy, with a map $\Delta \to \Delta_0$ that induces isomorphisms on the level of homology and homotopy groups after tensoring those of Δ with $\mathbb{Q}$. See [12] for a discussion of localization and rationalization of a simply connected CW-complex.

Conjecture 5.10. If $\Delta = \Delta(D)$ is the dual complex associated to a resolution of an isolated rational hypersurface singularity of dimension ≥ 3, then the rationalization of Δ is contractible.

We have that $\pi_1(\Delta) = 0$ from the above theorem. Conjecture 5.7 would give $H^i(\Delta, \mathbb{C}) = 0$ for $i > 0$, so the result would follow from the Hurewicz and Whitehead theorems, and the theory of localization.

5.3. **Examples.** We finish with some examples to illustrate the theory.

Example 5.11. Let $X = \mathbb{P}^n_{\mathbb{C}}$ be complex projective n-space and let $D = \sum_{i=1}^{n+1} H_i$, where the H_i are the coordinate hyperplanes. Then $\Delta(D) \approx S^{n-1}$, and we know that $H^q(D_{i_0\dots i_p}, \mathcal{O}_{D_{i_0\dots i_p}})$ is isomorphic to $\mathbb{C}$ when $q = 0$ and is 0 for $q > 0$, since $D_{i_0\dots i_p} \cong \mathbb{P}^{n-1-p}$. Then Theorem 5.1 (or Corollary 5.4) implies that $H^i(D, \mathcal{O}_D) \cong H^i(S^{n-1}, \mathbb{C})$, which we know is $\mathbb{C}$ for $i = 0, n-1$ and 0 else.

Example 5.12. Building on the previous example, let $X = X(\Sigma)$ be a smooth complex projective toric variety of dimension $n \geq 2$, and let $D = \sum D_i$ be the SNC sum of all the torus invariant divisors corresponding to edges of the fan Σ, so that $\omega_X \cong \mathcal{O}_X(-D)$. By Corollary 5.4 we have $H^0(\Delta(D), \mathbb{C}) \cong H^0(D, \mathcal{O}_D) \cong \mathbb{C}$ and we can compute the higher cohomologies of $\mathcal{O}_D$ by taking cohomology of the short exact sequence

$$0 \to \omega_X \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

and using that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ to conclude $H^i(\Delta(D), \mathbb{C}) \cong H^i(D, \mathcal{O}_D) \cong H^{i+1}(X, \omega_X) \cong H^{n-1-i}(X, \mathcal{O}_X)$ for $i > 0$, which is precisely $\mathbb{C}$ when $i = n-1$ and 0 else, confirming the observation that $\Delta(D) \approx S^{n-1}$ here as well.

Example 5.13 (Euler characteristic)**.** A consequence of the exact sequence of Lemma 3.1 on the level of Euler characteristics is

$$\chi(\mathcal{O}_D) = \sum_{p \geq 0} (-1)^p \Big(\sum_{i_0 < \dots < i_p} \chi(\mathcal{O}_{D_{i_0\dots i_p}}) \Big)$$

and we can calculate each $\chi(\mathcal{O}_{D_{i_0\dots i_p}})$ from the Hirzebruch-Riemann-Roch theorem. For instance, if $C = \sum_{i=1}^{N} C_i$ is a SNC curve on a surface X with each $C_i \cap C_j$ either empty or a single point for $i < j$, then the Riemann-Roch theorem gives us that

$$\chi(\mathcal{O}_{C_i}) = 1 - g(C_i).$$

Letting $e := |C_i \cap C_j \neq \varnothing|$ be the number of edges of $\Delta = \Delta(C)$ and noting N is the number of vertices of Δ, our formula above becomes

$$\chi(\mathcal{O}_C) = \sum_{i=1}^{N} \chi(\mathcal{O}_{C_i}) - \sum_{j=1}^{e} \chi(\mathcal{O}_{pt}) = N - e - \sum_{i=1}^{N} g(C_i) = \chi(\Delta) - \sum_{i=1}^{N} g(C_i)$$

which is a natural generalization to the SNC case.

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